

Crested Stationary Spectra of Weak Turbulence

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Two saturation mechanisms for unstable waves

□ Nonlinear mechanism

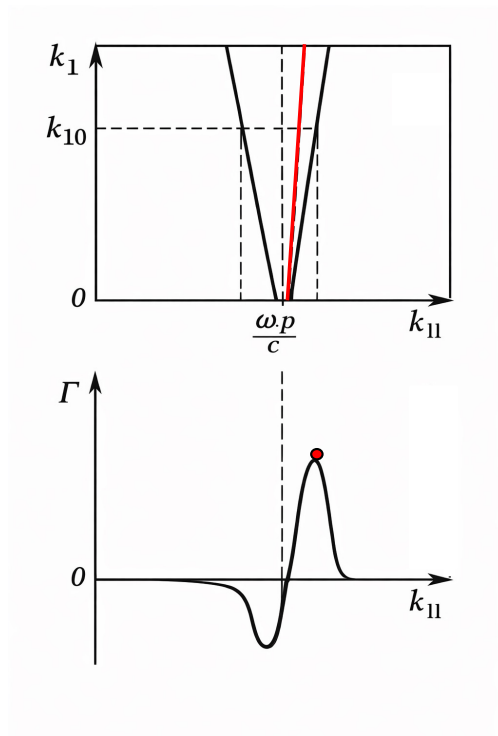
(nonlinear processes can remove waves from wave-particle resonance)

□ Quasilinear mechanism

(unstable waves can modify the drive)

Nonlinear saturation mechanism

The linear growth rate



□ Nonlinear scattering of waves balances the linear growth drive

$$\frac{\partial N(k; \theta; t)}{\partial t} = 2\Gamma(k; \theta)N + N \frac{\partial}{\partial k} \int k^2 Q(\theta; \theta') N(k; \theta') \sin \theta' d\theta'$$

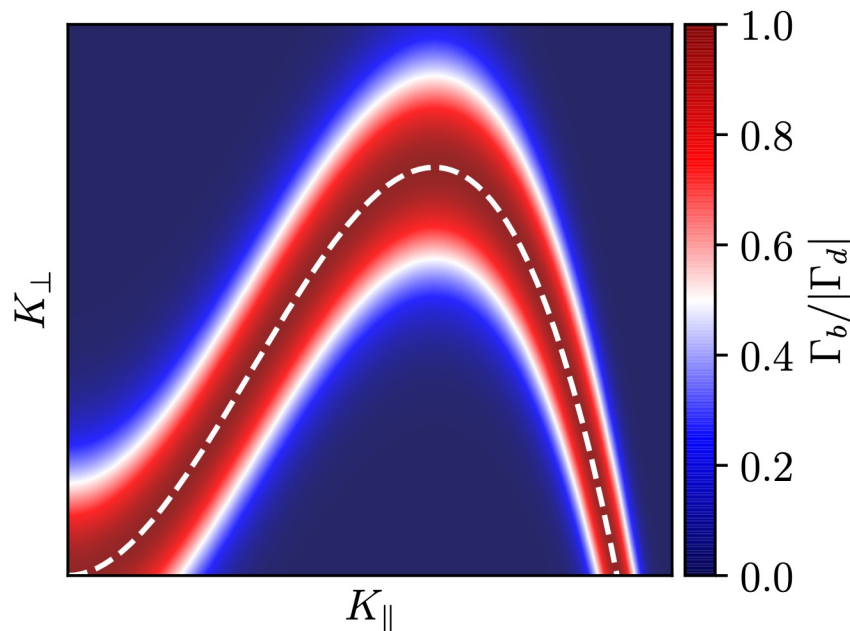
$$\begin{cases} 2\Gamma(k; \theta) + \frac{\partial}{\partial k} \int k^2 Q(\theta; \theta') N(k; \theta') \sin \theta' d\theta' = 0, & N \neq 0 \\ 2\Gamma(k; \theta) + \frac{\partial}{\partial k} \int k^2 Q(\theta; \theta') N(k; \theta') \sin \theta' d\theta' < 0, & N = 0 \end{cases}$$

- The scattering kernel $Q(\theta; \theta')$ is a third order polynomial in $\cos \theta$
- The linear growth rate has a peaked angular dependence
- A stationary spectrum must be located at the red marked crest

Quasilinear saturation mechanism

In a steady state, collisional damping or waves balances the modified kinetic drive

$$\Gamma_L (k_{\parallel}, k_{\perp}) = \Gamma_d (k_{\parallel}, k_{\perp})$$



*Growth-to-damping ratio
(cartoon)*

The marginal stability constraint selects a one-dimensional “crest” in the wave-vector space, where the waves reside

$$\frac{\partial}{\partial k_{\perp}} \left(\frac{\Gamma_L}{\Gamma_d} \right) = 0,$$
$$k_{\perp} = k_{\perp} (k_{\parallel})$$

Quasilinear constraint on runaway electron distribution

Wave-particle interaction for runaway electrons

The wave emission or absorption requires the following resonance condition:

$$\omega(\mathbf{k}) - k_{\parallel} v_{\parallel} = l \frac{|\omega_{ce}|}{\gamma}$$

The energy and momentum conservation laws show the dominant role of anomalous Doppler resonance ($l=-1$) for runaways with a small angular spread.

Low-frequency waves
change the electron
momentum more than
the electron energy



Enhanced pitch-angle scattering
of runaway electrons

Candidate waves

Whistlers

Compressional Alfvén waves (CAW)

Dispersion relation

$$\Omega = K |K_{\parallel}|$$

$$\Omega = \frac{1}{\sqrt{M}} K$$

Polarization

$$\mathbf{e} = \hat{x} + i \frac{|K_{\parallel}|}{K} \hat{y}$$

$$\mathbf{e} = \hat{x} + i \frac{1}{\sqrt{M}} \frac{K_{\perp}^2}{K^3} \hat{y}$$

Collisional damping rate

$$\Gamma_d = \nu K^2$$

$$\Gamma_d = \frac{\nu}{2} K^2$$

Applicability range:

$$\omega_{ci} \ll \omega \ll |\omega_{ce}|$$

Applicability range:

$$\omega \ll \omega_{ci}$$

Kinetic drive simplification

Kinetic drive from anomalous Doppler resonance:

$$\Gamma_b = \frac{\pi \Omega_{pe}^2 |1 + i e_y|^2}{2 \mathcal{E} \Omega} F(P_{\parallel})$$

$$F(P_{\parallel}) = 2\pi \int f(P_{\parallel}, P_{\perp}) P_{\perp} dP_{\perp}$$

is the distribution function integrated over the perpendicular momenta.

Assumptions:

$$P \gg 1$$

$$P_{\perp} \ll P_{\parallel}$$

$$P_{\parallel} \Leftrightarrow \frac{1}{\Omega_{pe} K_{\parallel}}$$

Growth-to-damping ratio is maximized when:

$$\frac{\partial}{\partial K_{\perp}} \frac{\Gamma_b}{\Gamma_d} = 0 \quad \longrightarrow \quad K_{\perp} = K_{\perp}(K_{\parallel})$$

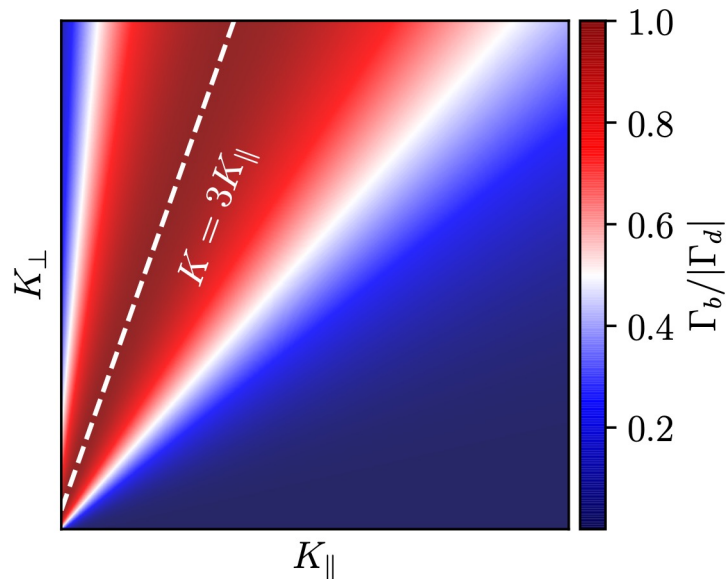
Constraint on the $P_{\parallel}$ distribution (1 of 2)

Whistlers

Growth-to-damping ratio for whistler waves is maximized at

$$K = 3 |K_{\parallel}|$$

The marginal stability constraint determines the integrated distribution function by setting the growth-to-damping ratio to unity



*Growth-to-damping ratio
(whistlers)*

$$\frac{\Gamma_b (K_{\parallel}, K_{\perp} (K_{\parallel}))}{\Gamma_d (K_{\parallel}, K_{\perp} (K_{\parallel}))} = 1$$



$$F (P_{\parallel}) = \frac{27\nu}{\pi\Omega_{pe}^2} \frac{1}{P_{\parallel}^2}$$

Constraint on the $P_{\parallel}$ distribution (2 of 2)

CAW

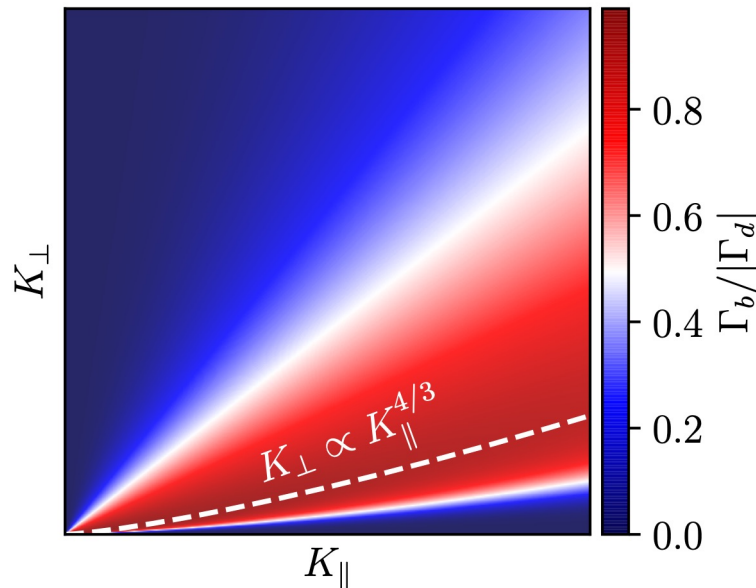
Growth-to-damping ratio for CAW waves is maximized at

$$\rightarrow K = |K_{\parallel}| \left[1 + \left(\frac{4}{3} M K_{\parallel}^2 \right)^{1/3} \right]^{1/2}$$

Distribution function integrated over transverse momenta:

$$F(P_{\parallel}) = \frac{2\nu\sqrt{M}}{\pi\Omega_{pe}^3} \frac{1}{P_{\parallel}^3}$$

The total energy of the REs converges at high momenta



Growth-to-damping ratio
(CAW)

**Self-consistent calculation
of runaway electron angular
spread
and wave spectrum**

Kinetic equation

$$\boxed{\hat{E} \frac{\partial f}{\partial P_{\parallel}}} - \boxed{\frac{1}{P_{\parallel}^2} \frac{\partial}{\partial P_{\parallel}} P_{\parallel}^2 \frac{P_{\perp}^2}{\hat{\tau}_{rad}} f - \frac{P_{\perp}}{P_{\parallel}} \frac{\partial}{\partial P_{\perp}} \frac{P_{\perp}^2}{\hat{\tau}_{rad}} f} = \boxed{\frac{1}{P_{\perp}} \frac{\partial}{\partial P_{\perp}} P_{\perp} D \frac{\partial f}{\partial P_{\perp}}}$$

- Electric field drive
- Synchrotron losses
- Elastic scattering

Assumptions:

$$P \gg 1$$

$$P_{\perp} \ll P_{\parallel}$$

Quasilinear diffusion operator:

$$D_{QL}(P_{\parallel}) = \frac{6\pi^3 \Omega_{pe}^2}{\hat{\tau}_{rad}} \frac{1}{P_{\parallel}^2} \frac{w_k(K_{\parallel}) K_{\perp} |1 + ie_y|^2}{\mathcal{E} \Omega^2}$$

High-Z scattering: $D_Z = \frac{Z+1}{2}$



Assumption:
 $D_{QL} \gg D_Z$

Wave spectrum and QL diffusion coefficient

The marginal stability constraint selects a crest, where the waves reside in the wave-vector space

Wave spectrum: $W_k = w_k (K_{\parallel}) \delta (K_{\perp} - K_{\perp} (K_{\parallel}))$

For a small angular spread of the RE beam, the diffusion coefficient is a function of parallel momentum:

$$D_{QL}(P_{\parallel}) = \frac{6\pi^3 \Omega_{pe}^2}{\hat{\tau}_{rad}} \frac{1}{P_{\parallel}^2} \frac{w_k(K_{\parallel}) K_{\perp} |1 + ie_y|^2}{\mathcal{E} \Omega^2}$$

Whistlers

$$D(P_{\parallel}) = \frac{8\sqrt{2}\pi^3}{3} \frac{\Omega_{pe}}{\hat{\tau}_{rad}} \frac{w_k(P_{\parallel})}{P_{\parallel}}$$

CAW

$$D(P_{\parallel}) = \frac{3\pi^3 \Omega_{pe}^{2/3}}{\hat{\tau}_{rad}} \left(\frac{4M}{3} \right)^{1/6} \frac{w_k}{P_{\parallel}^{4/3}}$$

Angular spread of the RE beam

To find the spread of RE in $P_{\perp}$ we seek a factorized form of the distribution function:

Ansatz for whistlers:

$$F(P_{\parallel}, P_{\perp}) = D_{QL} = \frac{27\nu}{\pi\Omega_{pe}^2} \frac{1}{P_{\parallel}^2} \frac{G(P_{\perp})}{2\pi \int_0^{\infty} P_{\perp} G(P_{\perp}) dP_{\perp}}$$

Reduced kinetic
equation:

$$\frac{P_{\parallel} D}{P_{\perp}} \frac{d}{dP_{\perp}} P_{\perp} \frac{dG}{dP_{\perp}} + P_{\perp} \frac{d}{dP_{\perp}} \frac{P_{\perp}^2}{\hat{\tau}_{rad}} G + 2\hat{E}G = 0$$

To satisfy the kinetic equation, the diffusion coefficient must be inversely proportional to the parallel momentum

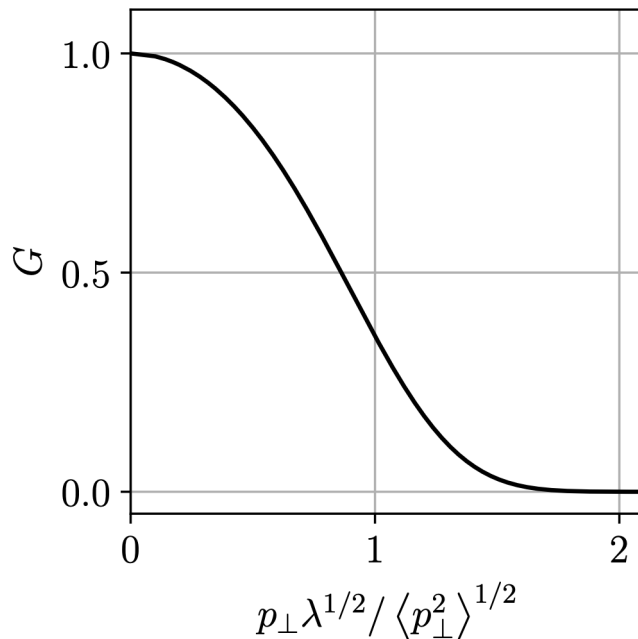
$$w_k(P_{\parallel}) = const \equiv w$$

$$D(P_{\parallel}) = \frac{D_0}{P_{\parallel}},$$



$$D_0 = \frac{8\sqrt{2}\pi^3}{3} \frac{\Omega_{pe} w}{\hat{\tau}_{rad}}$$

Eigenvalue problem for RE beam spread



$$\frac{d^2 G}{dx^2} + \left(x + \frac{1}{x} \right) \frac{dG}{dx} + \left(1 + \frac{\lambda}{x} \right) G = 0$$

$$x = \lambda \frac{P_{\perp}^2}{\langle P_{\perp}^2 \rangle} \quad \text{normalized perpendicular momentum}$$

$$\langle P_{\perp}^2 \rangle = \hat{E} \hat{\tau}_{rad}$$

The reduced kinetic equation has the following asymptotic solutions at large x

~~$$G_1 \sim \frac{A}{x},$$~~

$$G_2 \sim B \exp \left(-\frac{x^2}{2} \right)$$

Eigenvalue:

$$\lambda = 0.65$$

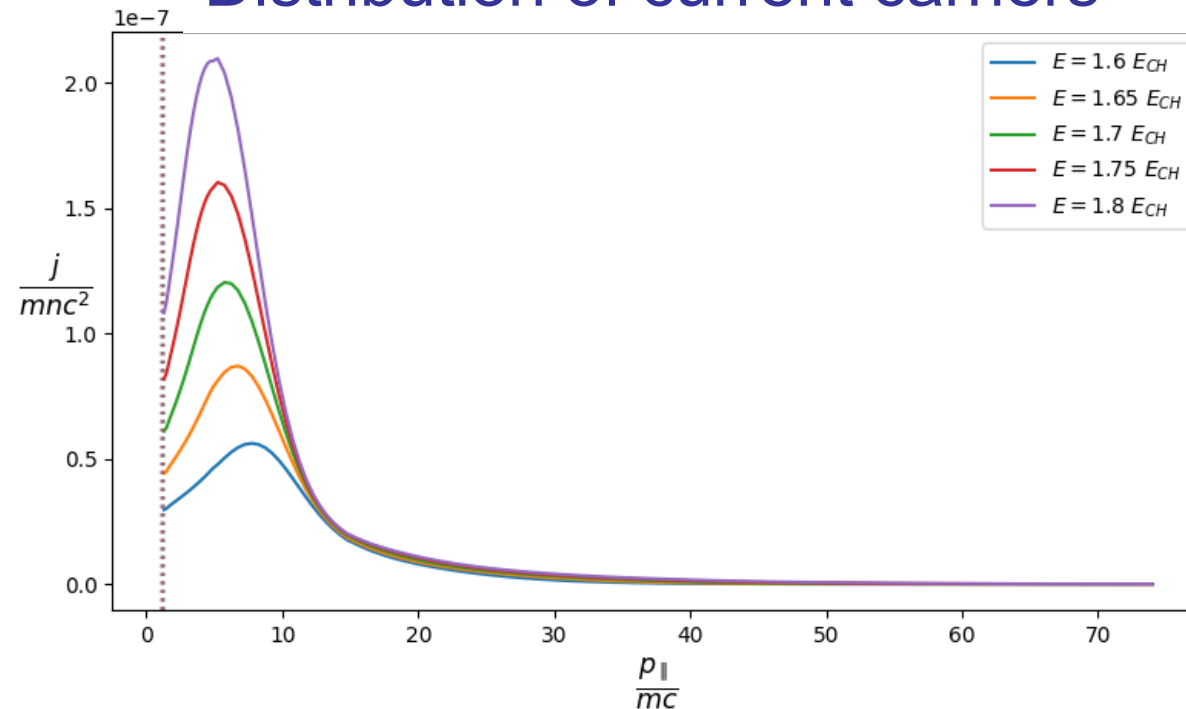
Crested wave spectrum in numerical model for RE

Tasks for numerical work

- ❑ Analyze the majority of current-carrying electrons (analytical theory describes only the RE tail)
- ❑ Describe the runaway production, the bulk plasma current and the evolution of the inductive field self-consistently
- ❑ Match the marginally stable and sub-threshold parts of the RE distribution.

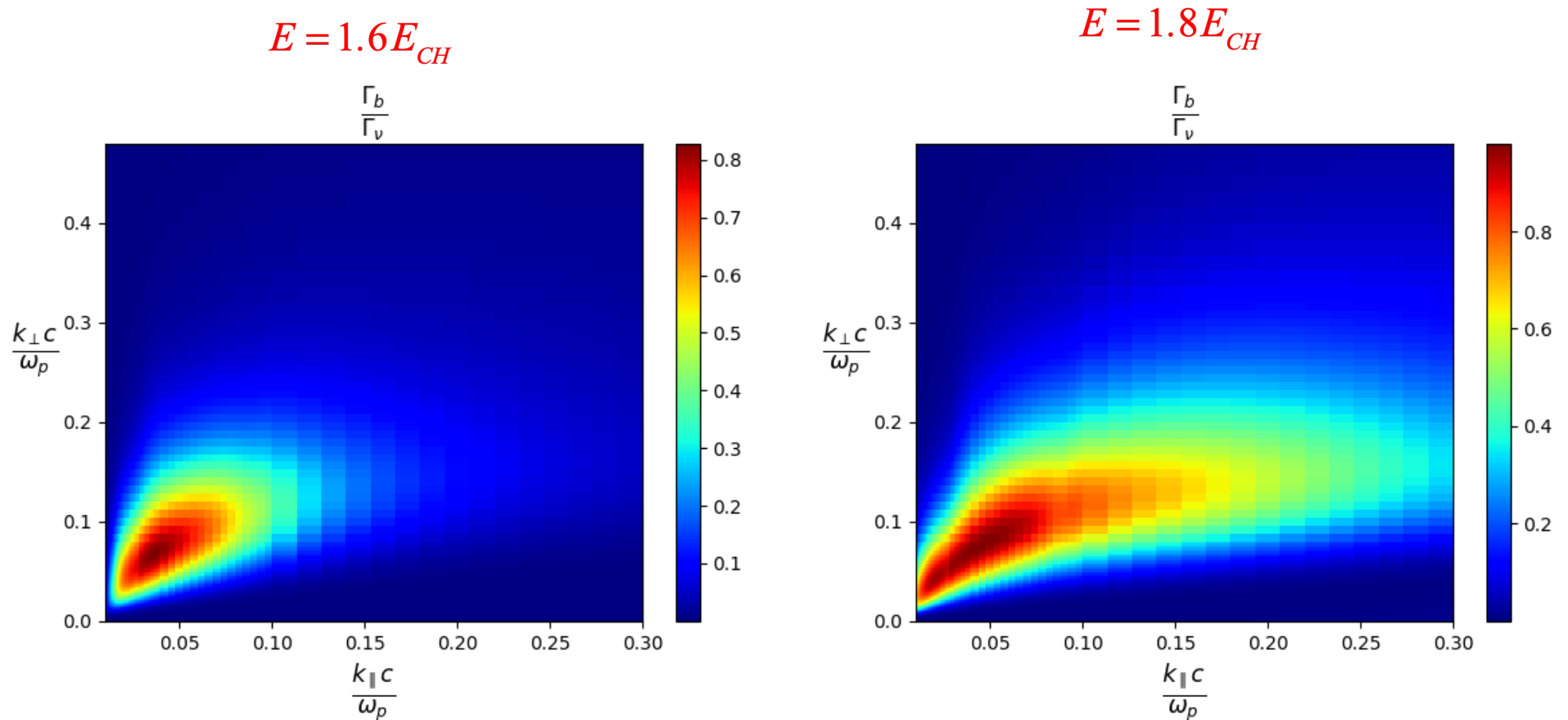
Runaway tail suppression by excited waves

Distribution of current carriers



- Most of the RE current comes from the sub-threshold part of the RE distribution

Wave spectrum dependence on the driving electric field



- ❑ Stronger electric field destabilizes shorter wavelengths that resonate with lower-energy electrons.

Conclusions

- ❑ Stationary spectra of weak turbulence tend to have a crested structure.
- ❑ The crested structure of stationary spectra reduces the dimensionality of the problem in analytical and numerical studies.
- ❑ Most of the current-carrying runaway electrons are only moderately relativistic in the wave-controlled state.

- [1] B.N. Breizman and D.I. Kiramov, *Marginal stability constraint on runaway electron distribution*, Physics of Plasmas **30**, 022301(2023).
- [2] B.N. Breizman, V.E Zakharov, and S.L. Musher, *Kinetics of Stimulated Scattering of Langmuir Waves by Plasma Ions*, Soviet Physics JETP **37**, 658 (1973).
- [3] K. Huang and B. N. Breizman, *The marginally stable state of runaway electrons controlled by self-excited waves*, 51st EPS Conference on Plasma Physics (July 7-11, 2025).
- [4] K. Huang and B. N. Breizman, *Runaway electron control by self-excited waves*, Submitted to Physics of Plasmas (2026)

Dimensionless notations

$D = m_e^{-1} c^{-2} v_{drag}^{-1} D$	Diffusion coefficient
$\hat{E} = E / E_c$	Electric field
$f = m_e^3 c^3 f / n$	Distribution function
$K = kc / \omega_{pe}$	Wave number
$M = m_i / m_e$	Ion-to-electron mass ratio
$P = pm_e^{-1} c^{-1}$	Particle momentum
$\nu = \nu_{ei} / \omega_{ce}$	Collision frequency
$W_k = W_k m_e^{-1} c^{-2}$	Spectral energy density
$\Gamma_b = \Gamma_b / \omega_{ce}$	Kinetic growth rate
$\hat{\tau}_{rad} = v_{drag} \tau_{rad}$	Synchrotron time
$\Omega = \omega / \omega_{ce}$	Wave frequency
$\Omega_{pe} = \omega_{pe} / \omega_{ce}$	Plasma frequency
$\mathcal{E} =$	Normalized wave energy