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Two-dimensional kinetic full wave analysis of O-X-B mode conversion in tokamak plasmas

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- Kinetic full wave analysis using integral form of $\overleftrightarrow{\epsilon}$
- O-X-B mode conversion of EC waves in tokamak plasmas
- Two-dimensional analysis of O-X-B mode conversion
- Kinetic full wave analysis of inhomogeneous ECR
- Summary

Kinetic full wave analysis in an inhomogeneous plasma

- **Motivation of kinetic full wave analysis**
  - **Description of waves with short wave length**
    - Bernstein waves, contribution of energetic particles
  - **Inhomogeneous magnetic field along the field line**
    - Absorption near cyclotron resonance
  - **Kinetic full wave analysis using FEM without iteration**
    - Wave numbers are not determined a priori
- **Previous kinetic full wave analyses**
  - **Cold plasma wave number approach**: no kinetic waves
  - **Differential operator approach**: $\boldsymbol{k}$ replaced by $\mathrm{i}\,\nabla$; up to 2nd-order
  - **Spectrum approach**: Fourier expansion; large numerical resources
- **Integral operator approach**:
  - Integral form of dielectric tensor $\int \epsilon(x - x') \cdot E(x')\,\mathrm{d}x'$

Derivation of integral form of dielectric tensor

- **Uniform plasma**:
  - **Particle orbit**: $z = z' + v_z(t - t')$, **Variable transformation**: $v_z = \dfrac{z - z'}{t - t'}$
- **Perturbed velocity distribution function with $\boldsymbol{E}(z)\,\mathrm{e}^{-\mathrm{i}\,\omega t}$**: $\tau = t - t'$

$$f(z, \boldsymbol{v}) = \frac{n}{(2\pi T/m)^{3/2}} \frac{q}{T} \int_0^\infty \mathrm{d}\tau\, \boldsymbol{v} \cdot \boldsymbol{E}(z') \exp\left[-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2T} + \mathrm{i}\,\omega\tau \right]$$

- **Induced current density**: variable transformation $v_z \;\Rightarrow\; z'$

$$\boldsymbol{J}(z) = q \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f(z, \boldsymbol{v}) = \int \mathrm{d}z'\, \overleftrightarrow{\sigma}(z - z') \cdot \boldsymbol{E}(z')$$

- **Electric conductivity tensor**: e.g. zz component

$$\sigma_{zz}(z - z') = \frac{nq^2}{\sqrt{2\pi}\, m v_{\mathrm{T}}^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}z'}{\tau} \int_0^\infty \mathrm{d}\tau \frac{(z - z')^2}{\tau^2} \exp\left[-\frac{1}{2}\frac{(z - z')^2}{v_{\mathrm{T}}^2 \tau^2} + \mathrm{i}\,\omega\tau \right]$$

Integral form of dielectric tensor and kernel function

- **Kernel function**: Plasma dispersion kernel function

$$U_n(\xi,\eta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}\hat{\tau}\, \hat{\tau}^{n-1} \exp\left[-\frac{1}{2}\frac{\xi^2}{\hat{\tau}^2} - \frac{1}{2}\eta^2\hat{\tau}^2 + \mathrm{i}\,\hat{\tau} \right]$$

- **Conductivity tensor**: e.g. xx component

$$\sigma_{xx}(x-x') = \frac{nq^2}{m\omega} \int_{-\infty}^{\infty} \mathrm{d}\hat{x}'\; \xi^2\, U_{-2}(\xi,0)$$

$$\hat{x} = \frac{\omega x}{v_{\mathrm{T}}}, \qquad \hat{\tau} = \omega\tau, \qquad \xi = \frac{\omega(x-x')}{v_{\mathrm{T}}}$$

- **Fourier transform of U_n**:

$$V_n(\hat{k}) = \int_{-\infty}^{\infty} \mathrm{d}\xi\; \mathrm{e}^{-\mathrm{i}\,\hat{k}\xi}\, U_n(\xi,0) = \frac{1}{\mathrm{i}\sqrt{\pi}} \frac{1}{\hat{k}} \int_{-\infty}^{\infty} \mathrm{d}u\, \frac{u^n}{u-(1/\hat{k})}\, \mathrm{e}^{-u^2/2}$$

  – $V_0(\hat{k})$ corresponds to the plasma dispersion function $Z(1/\hat{k})$

Plasma dispersion kernel function

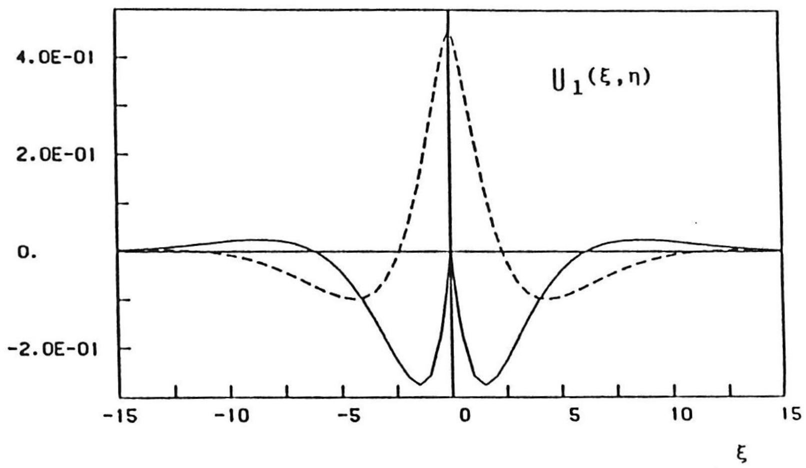
- **A general form of kernel function**: non-zero k_y, acceleration $\Rightarrow \eta$

$$U_n(\xi,\eta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}\tau\, \tau^{n-1} \exp\left[-\frac{1}{2}\frac{\xi^2}{\tau^2} - \frac{1}{2}\eta^2\tau^2 + \mathrm{i}\,\tau\right]$$

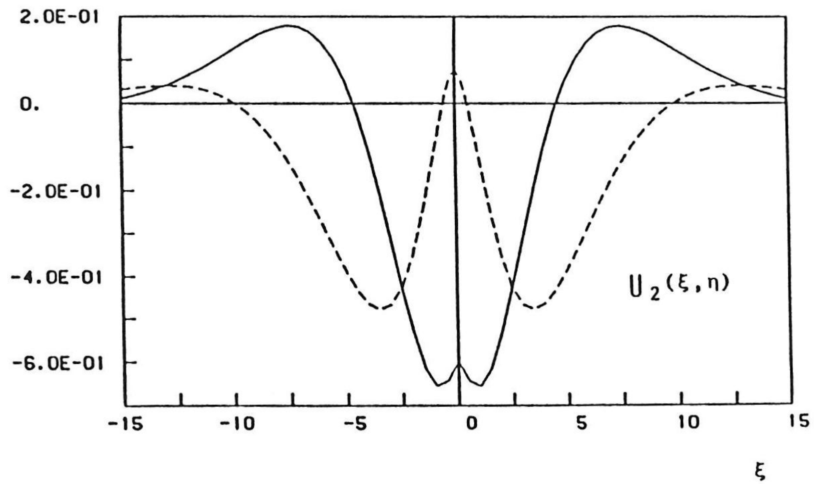
- **Properties of U_n**
  - **Derivative and partial integral**:

$$\frac{\partial U_n}{\partial \xi} = -\xi U_{n-2}, \qquad \mathrm{i}\,U_n + nU_{n-1} + \frac{\partial^2}{\partial \xi^2}U_{n+1} - \eta^2 U_{n+1} = \begin{cases} -\delta(\xi) & \text{for } n=0 \\ -1 & \text{for } n=1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

$U_1(\xi,0)$

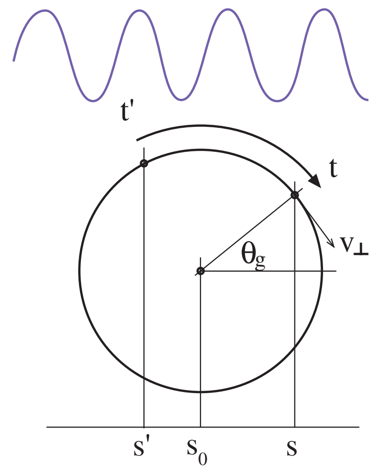


$U_2(\xi,0)$



Finite Larmor radius effects

- **Transformation of Integral Variables**
  - Transformation from the velocity space variables $(v_\perp, \theta_g)$ to the particle position s' and the guiding center position s_0.



  - Jacobian: $J = \dfrac{\partial(v_\perp, \theta_g)}{\partial(s', s_0)} = -\dfrac{\omega_c^2}{v_\perp \sin \omega_c \tau}$.
  - Express $v_\perp$ and θ_g by s' and s_0 using $\tau = t - t'$, e.g.,

$$v_\perp \sin(\omega_c \tau + \theta_g) = \frac{\omega_c}{v_\perp} \frac{s - s'}{2} \frac{1}{\tan \frac{1}{2}\omega_c \tau} + \frac{\omega_c}{v_\perp}\left(\frac{s + s'}{2} - s_0\right) \tan \frac{1}{2}\omega_c \tau$$

- **Integration over τ**: Fourier expansion with cyclotron motion
- **Integration over $v_\parallel$**: Plasma dispersion function
- **Conductivity tensor**: (ℓ : cyclotron harmonics number)

$$\overleftrightarrow{\sigma}(s, s', \chi_0, \zeta_0) = -i n_0 \frac{q^2}{m} \sum_\ell \int \mathrm{d}s_0 \, \overleftrightarrow{H}_\ell(s - s_0, s' - s_0; s_0, \chi_0, \zeta_0)$$

Plasma gyro kernel function

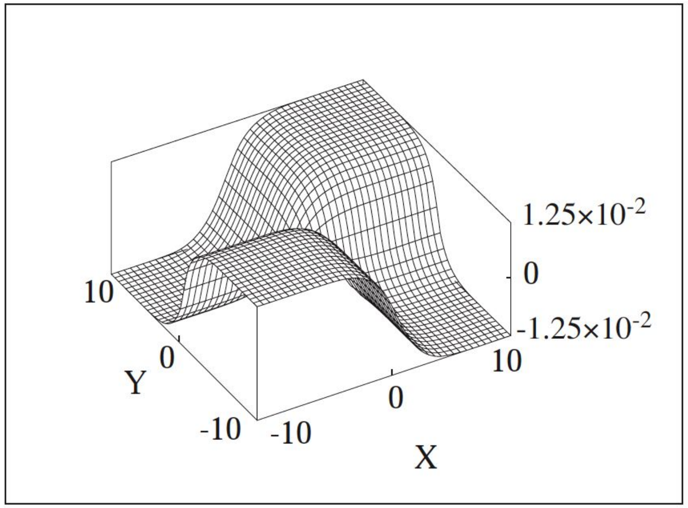
- **Kernel function for gyro motion and its integral:**

$$F_n^{(i)}(X,Y) \equiv \frac{1}{2\pi^2}\int_0^{\pi} \mathrm{d}\theta \exp\left[-\frac{X^2}{1+\cos\theta} - \frac{Y^2}{1-\cos\theta}\right] f_n^{(i)}(\theta)$$

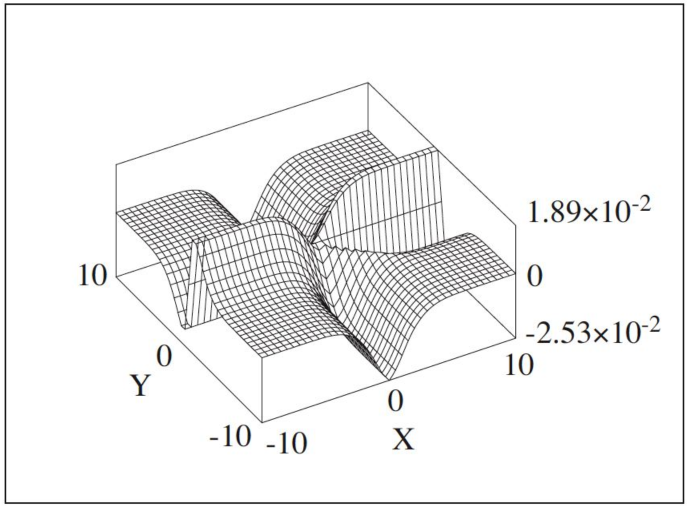
$$\mathcal{F}_n^{(ijk)}(X,Y) \equiv \int_0^{Y} \mathrm{d}Y' \int_0^{X+Y'} \mathrm{d}X' X'^j Y'^k F_n^{(i)}(X',Y')$$

$$f_n^{(i)}(\theta) = \begin{cases} \dfrac{\cos n\theta}{\sin\theta} & (i=1) \\ \sin n\theta & (i=2) \\ \dfrac{\sin n\theta}{\sin^2\theta} & (i=3) \\ \dfrac{\cos\theta \sin n\theta}{\sin^2\theta} & (i=4) \end{cases}$$

$\mathcal{F}_{\prime}^{(\infty\prime\prime)}$



$\mathcal{F}_{\infty}^{(\infty\prime\prime)}$



O-X-B mode conversion of electron cyclotron waves

- **EC heating and current drive in over-dense plasmas**
  - Experimental observation of EC H&CD in a high-density plasma above the cutoff density
  - Possible mechanism is the mode-conversion to the electron Bernstein waves (EBW)
- **O-X-B mode conversion**
  - **HFS excitation**: X-mode is converted to EBW near the UHR
  - **LFS excitation**: Cutoff layer exists for both O and X modes
  - For **optimum injection angle** derived by Hansen et al. [PPCF, **27** 1077 (1985)

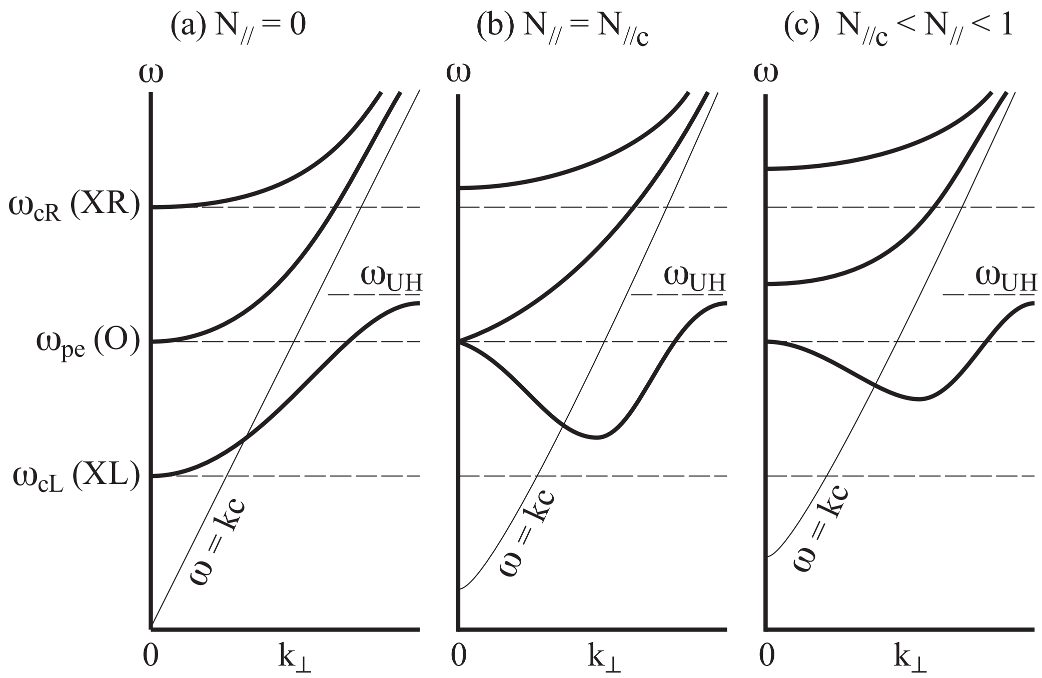
    $$N_{\parallel}^2 = \frac{|\omega_{\rm ce}|}{\omega + |\omega_{\rm ce}|},$$

    mode-conversion from O-mode to X mode occurs.
  - The X-mode is converted to EBW near the UHR, and the EBW is absorbed by EC damping near ECR.

O-X-B mode conversion of EC waves in tokamak

- **For the optimum injection angle,**
  - O-mode cutoff and X-mode cutoff are located at the same position.
  - O-X mode conversion: k changes the sign, forward to backward
- **When the injection angle is not optimum,**
  - Evanescent layer appears between the O and X cutoffs.
  - Geometrical optics cannot describe tunneling.

Dispersion relation for fixed $N_{||}$



$k_{\|}$ dependence of dispersion relation

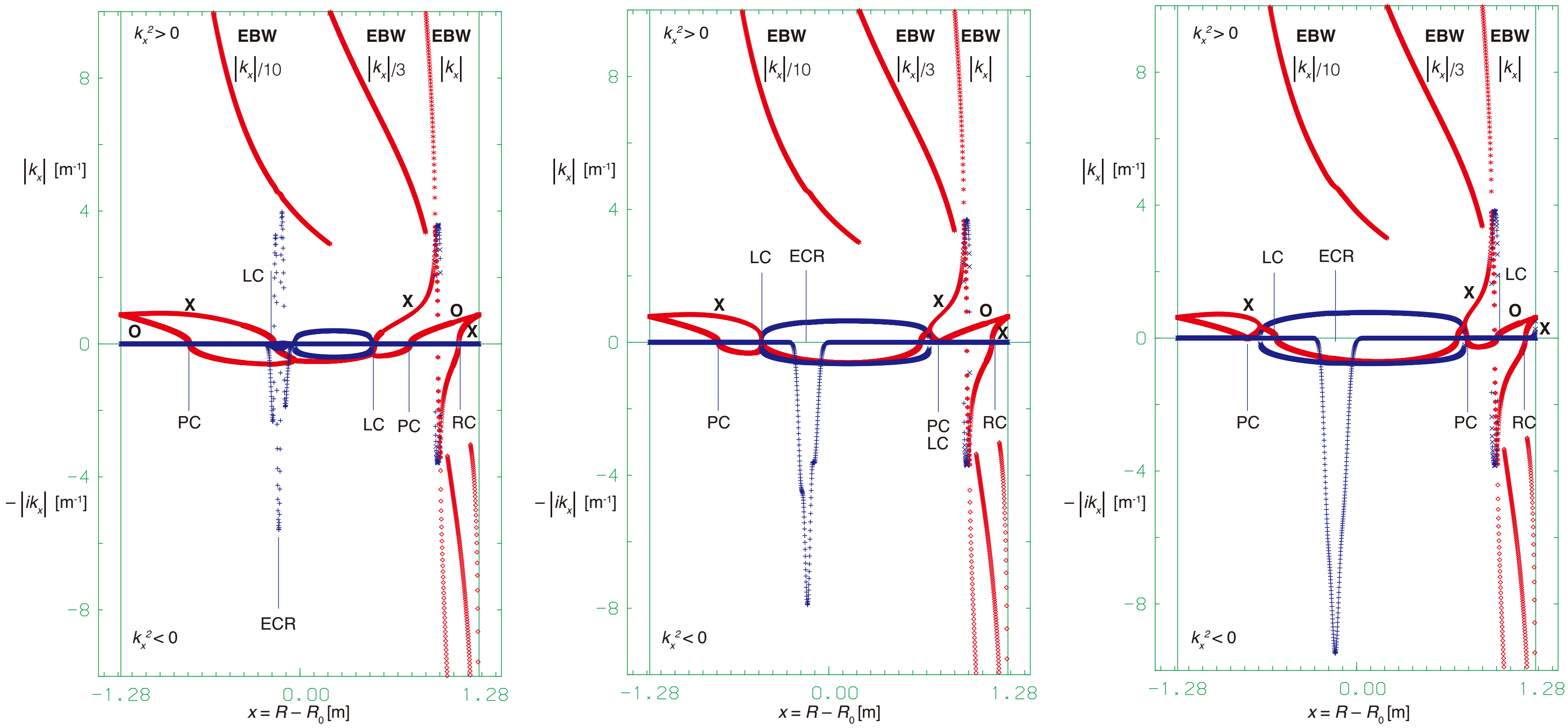
Four times enlarged parameters of the spherical tokamak LATE

$R_0 = 1.76\,\mathrm{m}$, $a = 1.28\,\mathrm{m}$, $B_0 = 0.08\,\mathrm{T}$ $n_e(0) = 1.2 \times 10^{17}\,\mathrm{m}^{-3}$, $f = 2.45\,\mathrm{GHz}$

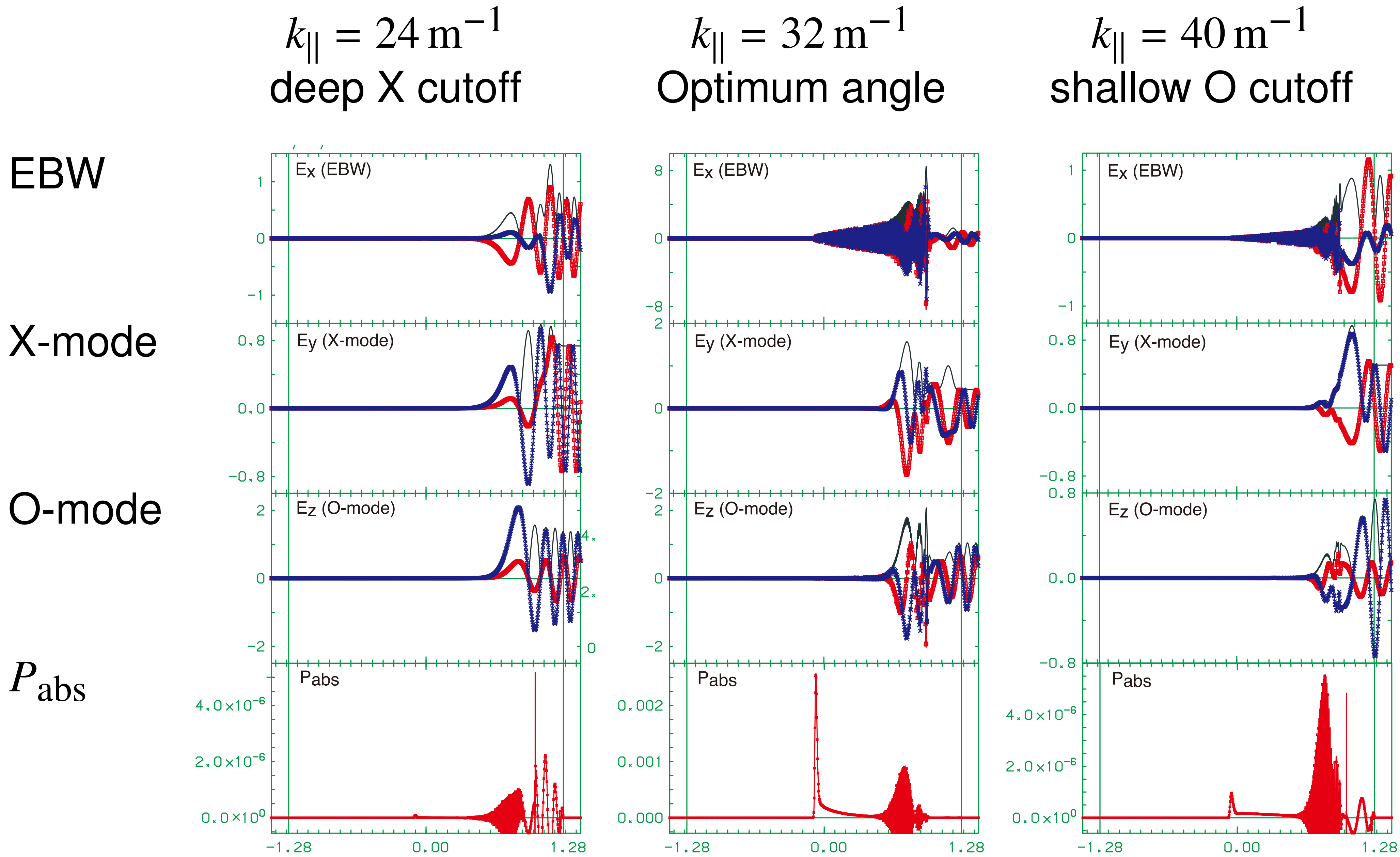
$k_{\|} = 24\,\mathrm{m}^{-1}$ deep X cutoff

$k_{\|} = 32\,\mathrm{m}^{-1}$ Optimum angle

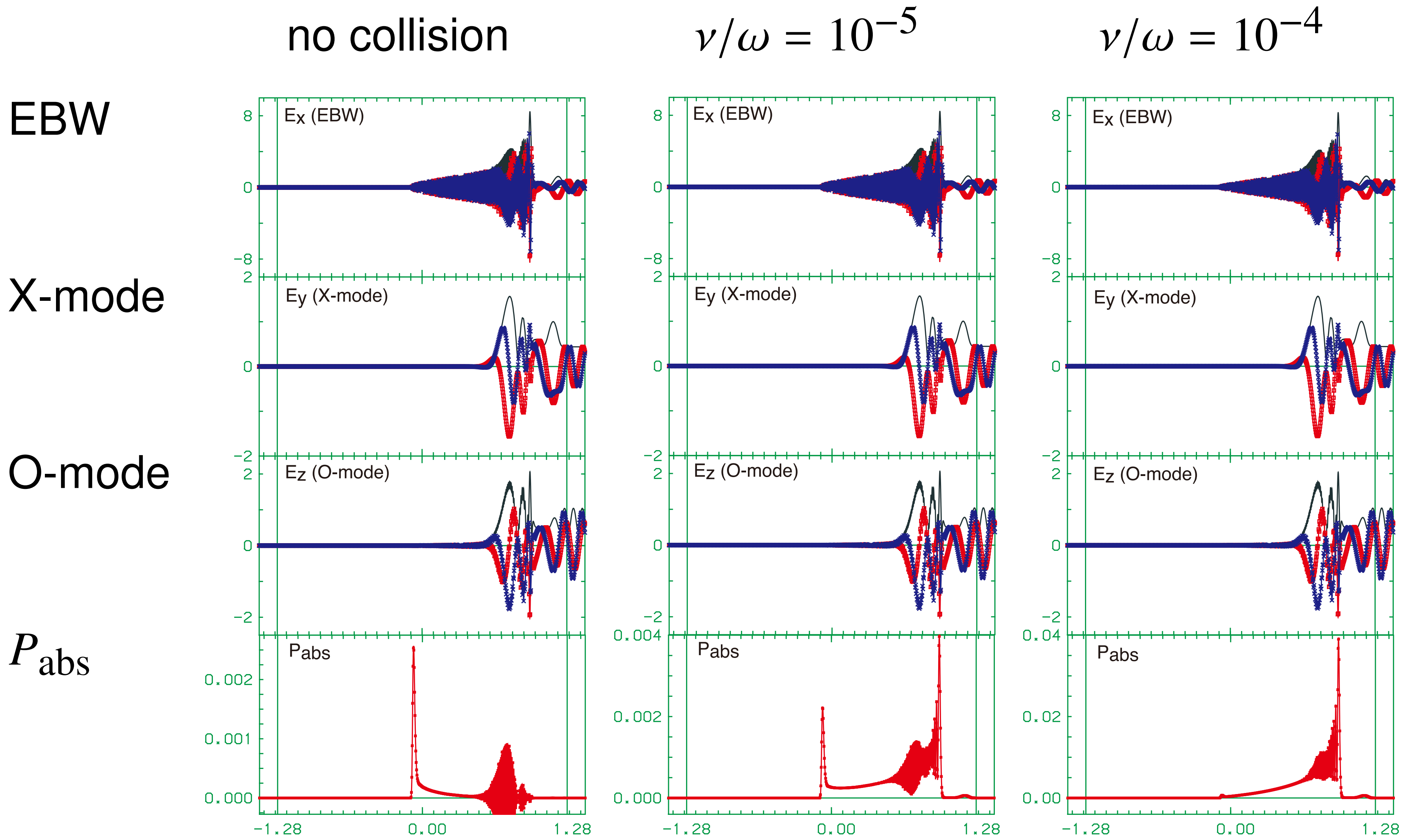
$k_{\|} = 40\,\mathrm{m}^{-1}$ shallow O cutoff



1D kinetic full wave analysis using integral form of $\overleftrightarrow{\epsilon}$



Effect of collisional damping

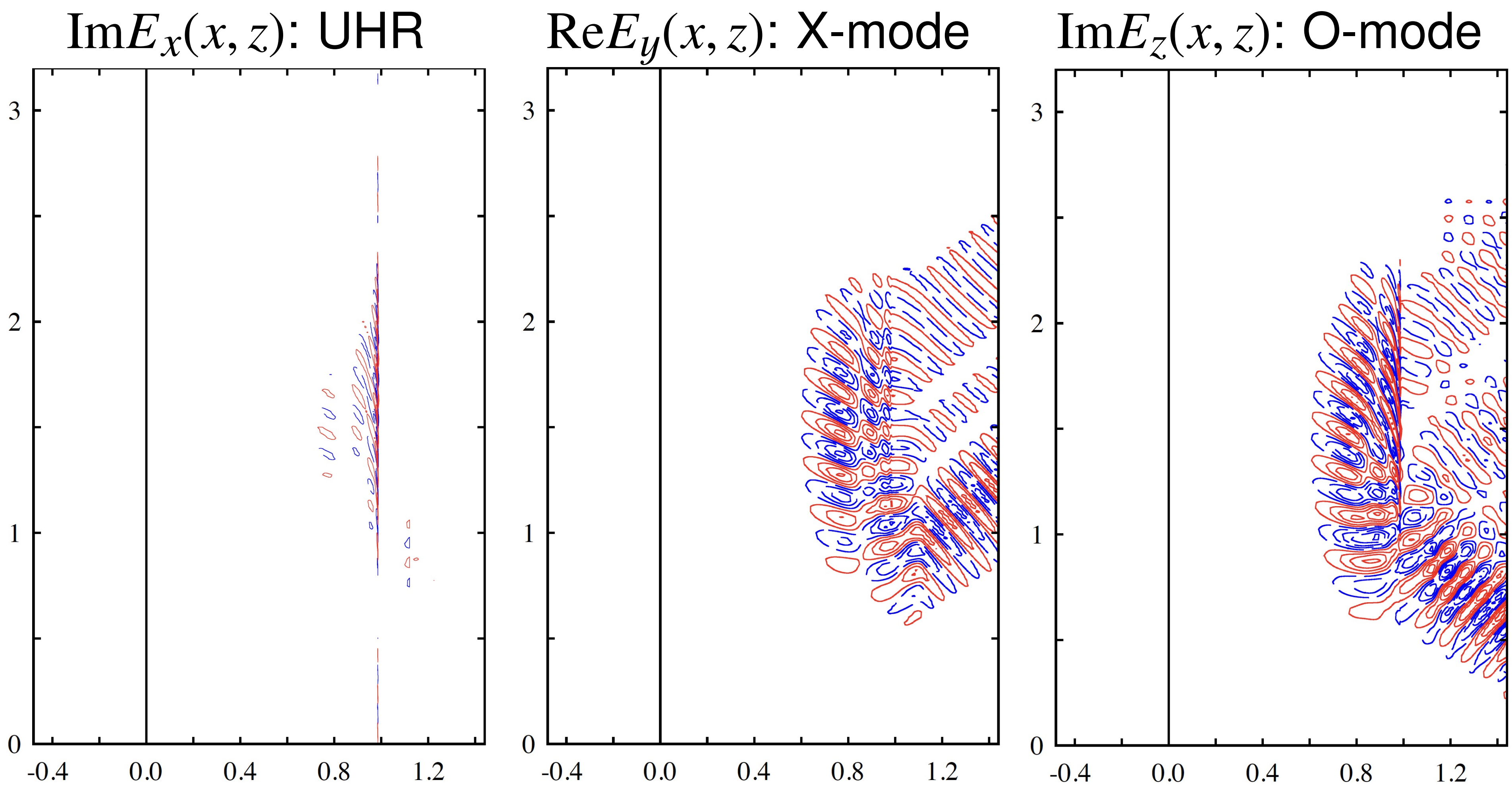


2D analysis of O-X mode conversion

Cold plasma model

Mode conversion of O-mode excited by WG antenna

Collisional damping of mode converted X-mode near UHR



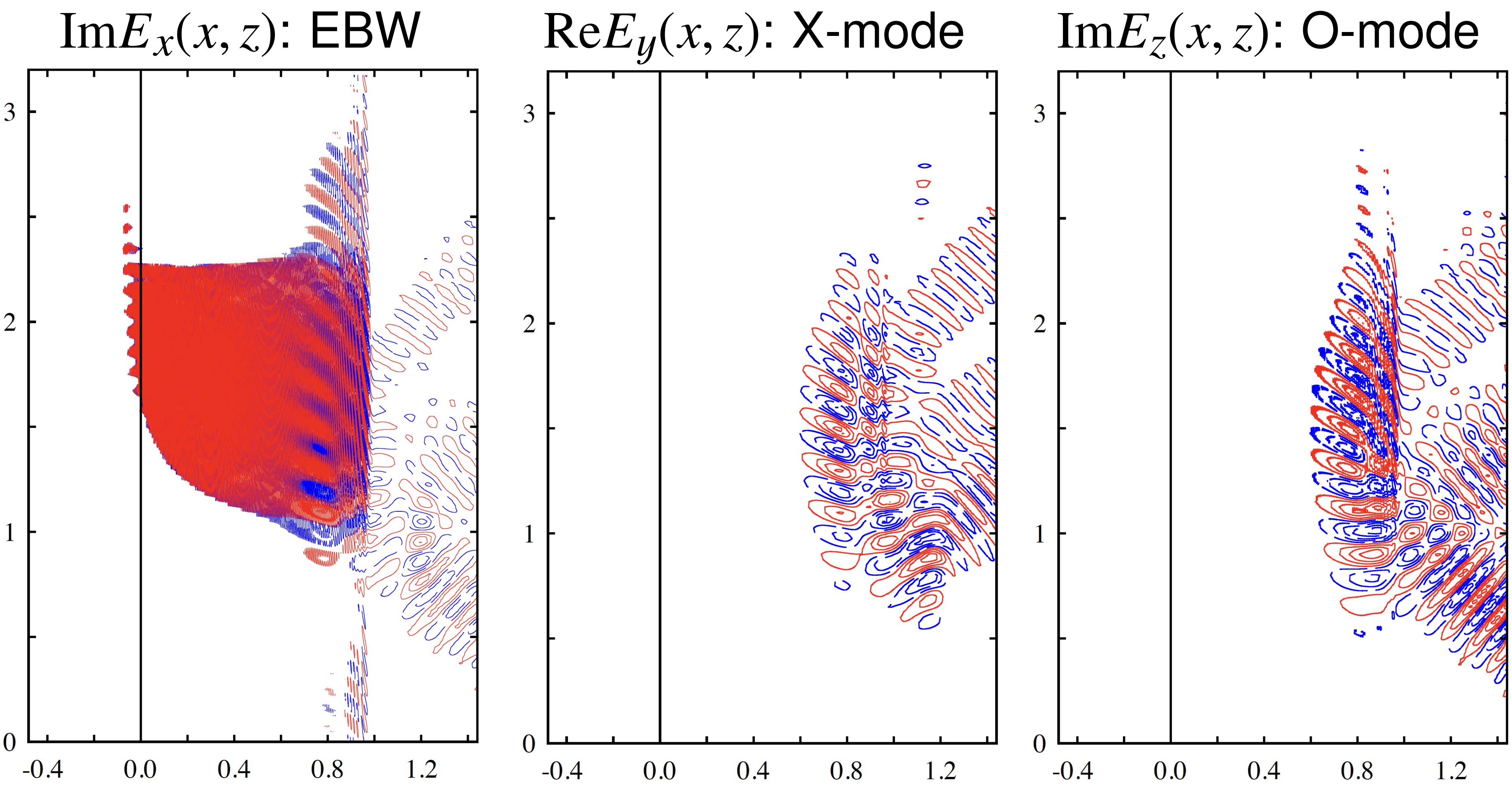
2D analysis of O-X-B mode conversion

Kinetic plasma model using the integral form of dielectric tensor

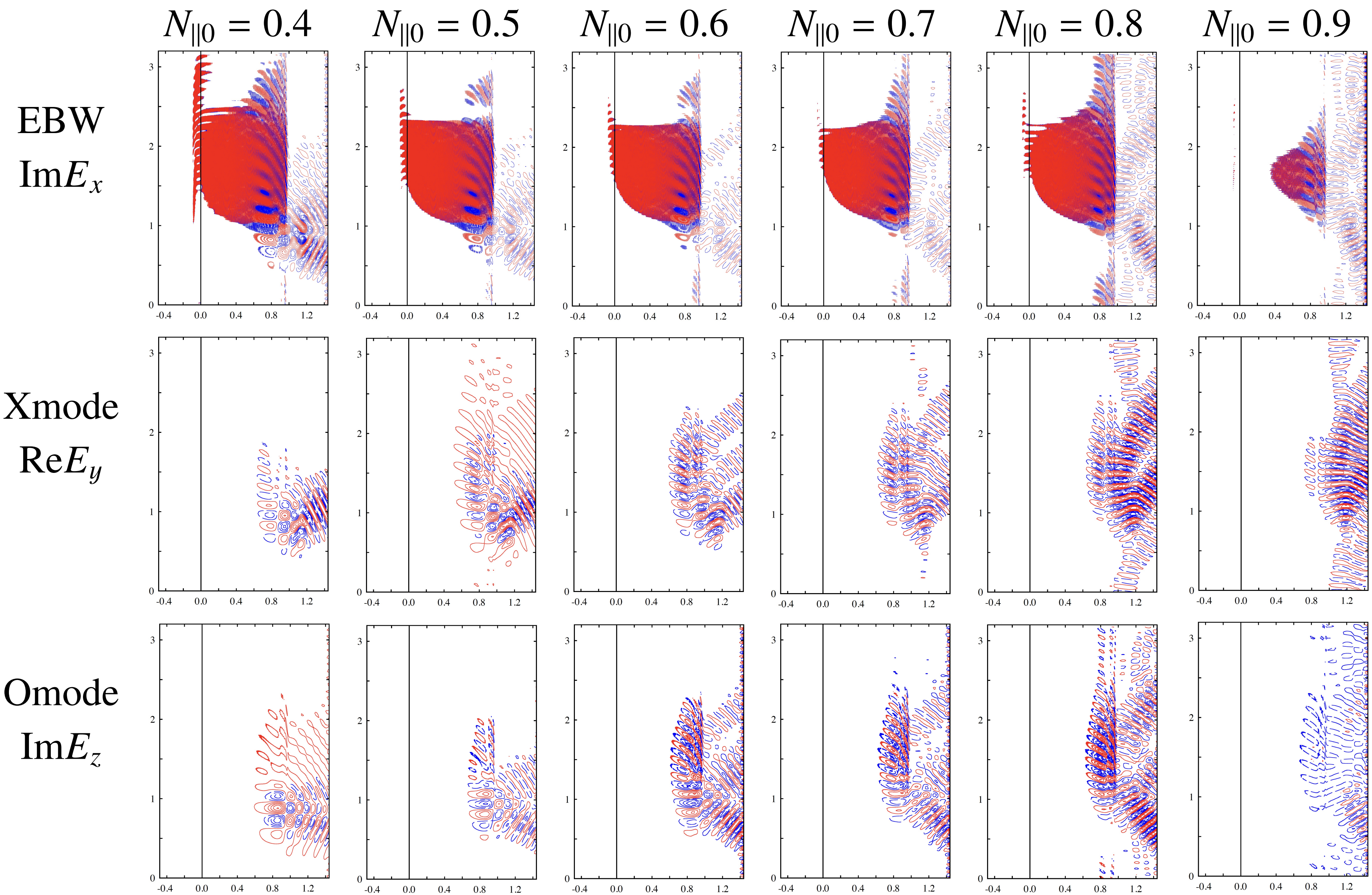
Mode conversion of O-mode excited by WG antenna

Mode conversion from X-mode to EBW near UHR

EC damping of EBW near ECR



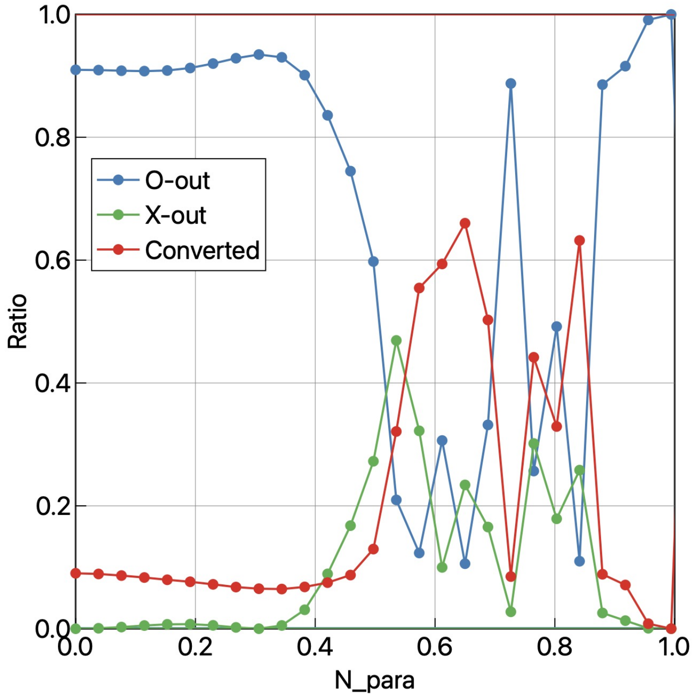
Dependence on injection angle



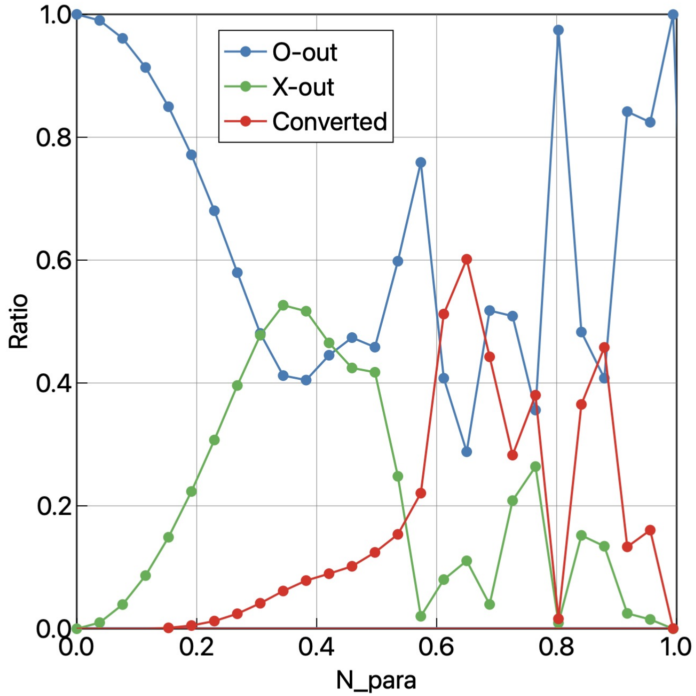
$N_{\parallel}$ dependence of absorption and reflection coefficients

Absorption (Abs) and Reflection (O_R, X_R) coefficients

Cold plasma model



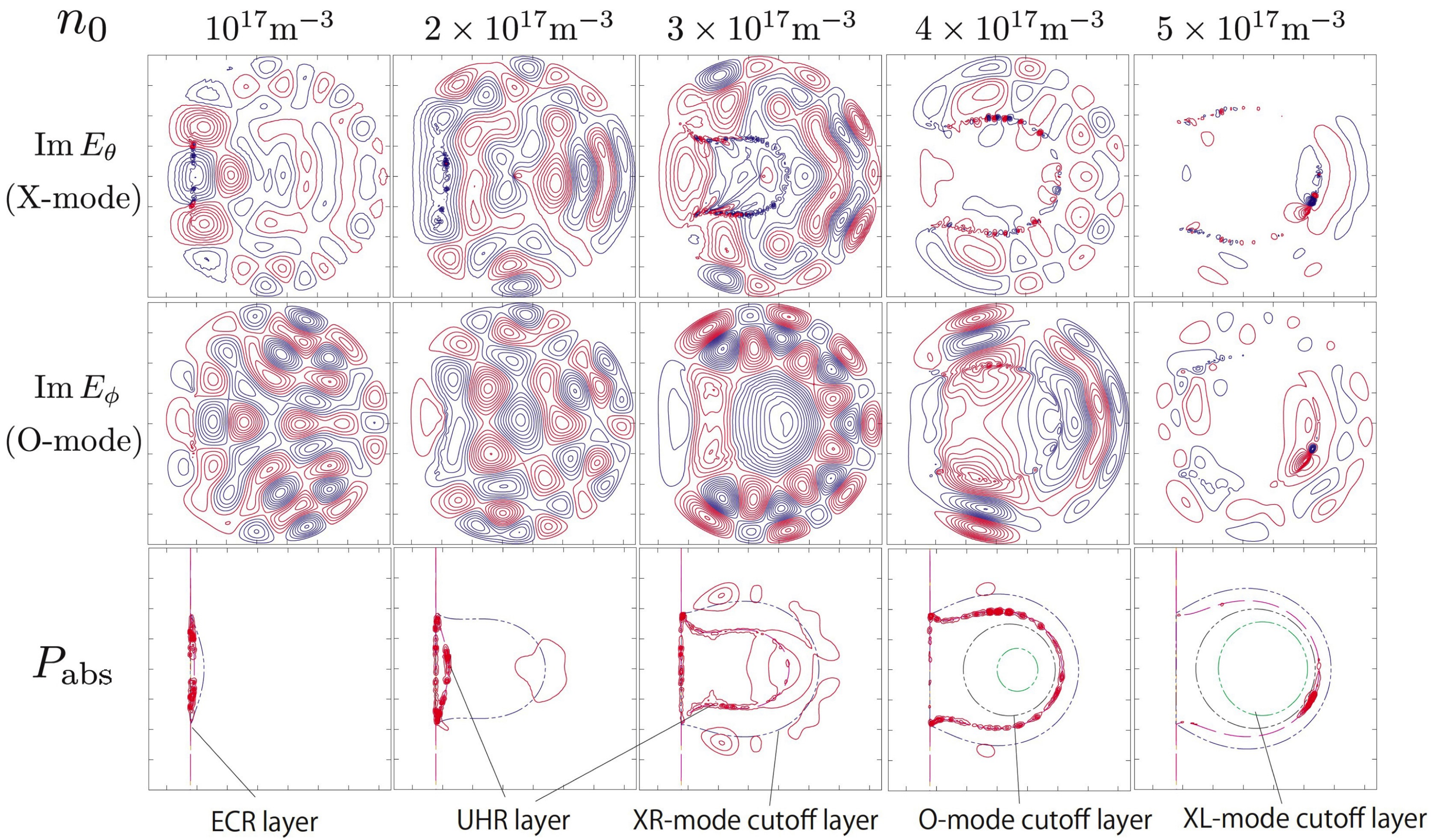
Kinetic plasma model



Conversion rate to EBW has maximum near the optimum injection angle ($\sim 32\,\mathrm{m}^{-1}$, $N_{\parallel} \sim= 0.64$).
Standing wave formation may affect the conversion rate.

2D analysis in poloidal cross section (cold plasma model)

$R = 0.22\,\mathrm{m},\ a = 0.16\,\mathrm{m},\ B_0 = 0.072\,\mathrm{T},\ f = 5\,\mathrm{GHz},\ n_\phi = 24$



Non-uniform magnetic field strength along the field line

Integral form of $\overleftrightarrow{\epsilon}$ in non-uniform magnetic field

- **First-order non-uniformity**

$$B(z) = B_0 \left(1 + \frac{z}{L_1} \right)$$

- **Adiabatic motion**: $\epsilon = mv^2/2$, $\mu = mv_\perp^2/2B$

$$z' - z = -\sqrt{\frac{2}{m}} \sqrt{\epsilon - \mu B_0 \left(1 + \frac{z}{L_1} \right)}\, \tau - \frac{\mu B_0}{2mL_1} \tau^2$$

- **Integral over ϵ is replaced by integral over z'**

$$\epsilon = \mu B_0 \left(1 + \frac{z + z'}{2L_1} \right) + \frac{m(z - z')^2}{2\tau^2} + \frac{\mu^2 B_0^2}{8mL_1^2} \tau^2$$

- **Integral over μ is approximated by** : $\mu_c = 2\sqrt{2mT} L_1 / B_0 \tau$

$$\int_0^\infty \mathrm{d}\mu\, f(\mu) \exp^{-\mu^2/\mu_c^2} \sim \int_0^{\mu_c} \mathrm{d}\mu\, f(\mu)$$

Integral form dielectric tensor in a mirror field

- **Dielectric tensor**

$$\overleftrightarrow{\epsilon}(z,z') = \delta(z-z')\overleftrightarrow{I} + \frac{\omega_{p0}^2}{\omega^2}\begin{pmatrix} (\chi_+ + \chi_-)/2 & -\mathrm{i}(\chi_+ - \chi_-)/2 & 0 \\ \mathrm{i}(\chi_+ - \chi_-)/2 & (\chi_+ + \chi_-)/2 & 0 \\ 0 & 0 & \chi_0 \end{pmatrix}$$

- **Components of dielectric tensor** expressed with kernel functions

$$\chi_\pm = \frac{(1+\kappa z)^{3/2}(1+\kappa z')^{3/2}}{(1+\kappa(z+z')/2)^2} U_0(\xi_\pm, 0)$$

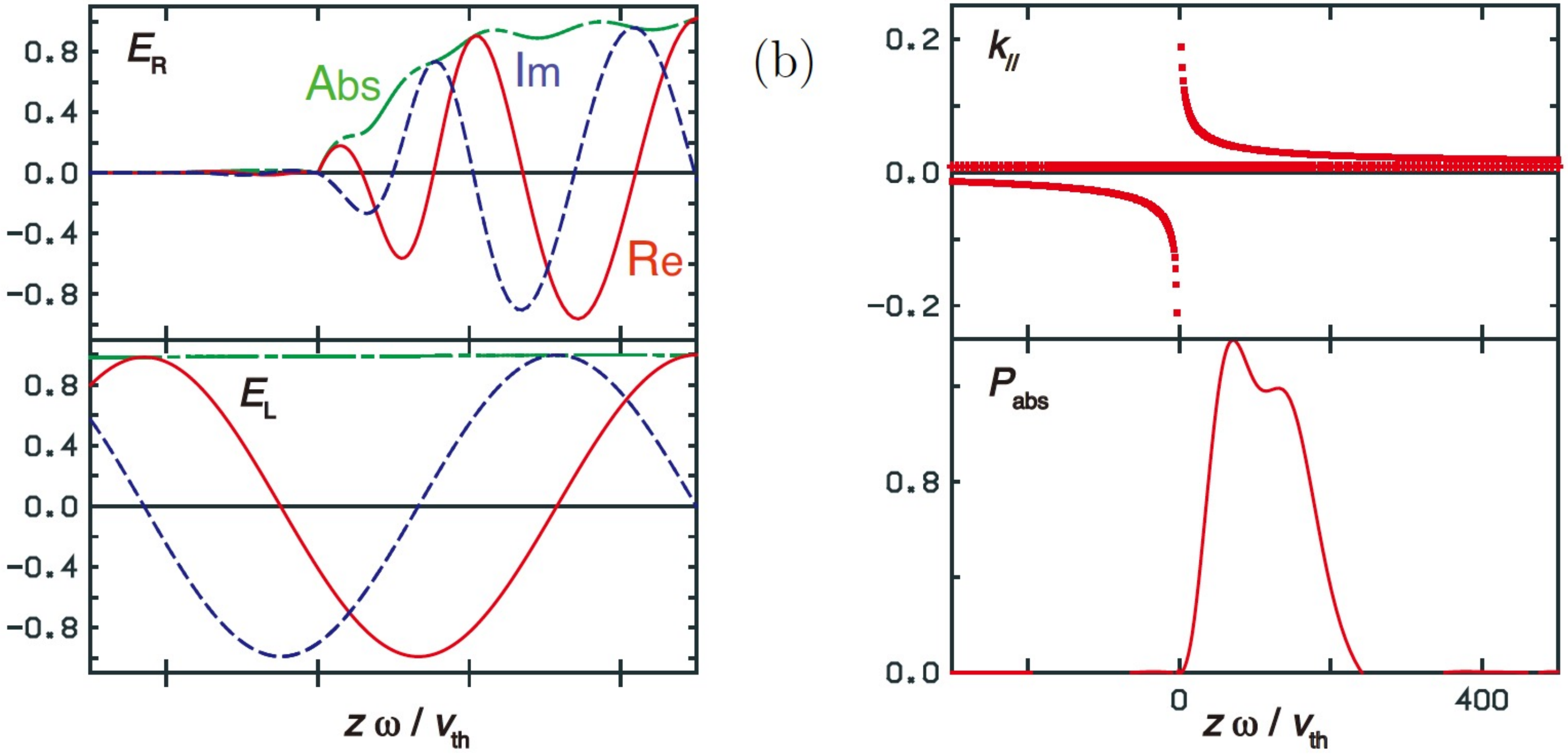
$$\chi_0 = \frac{(1+\kappa z)(1+\kappa z')}{(1+\kappa(z+z')/2)}\left[\xi U_{-2}(\xi,0) - \frac{\kappa^2}{2(1+\kappa(z+z')/2)^2} U_2(\xi,0)\right]$$

$$\xi = \frac{\omega(z-z')}{v_T}, \qquad \xi_\pm = \frac{(\omega\pm\Omega)(z-z')}{v_T}$$

$$\Omega = \frac{qB_0}{m}\left(1+\frac{z+z'}{2L}\right), \qquad \kappa = \frac{v_T}{\omega L}$$

Magnetic beach heating at ECR

- **High-Field-Side excitation**: $\omega_{\rm pe}^2/\omega_{\rm ce}^2 = 0.5$, $\beta = v_{\rm the}/c = 0.01$
- **Right-hand-circularly-polarized wave** :
  - absorbed at the cyclotron resonance
- **Left-hand-circularly-polarized wave** :
  - transmitted without absorption



Integral form of $\overleftrightarrow{\epsilon}$ in non-uniform magnetic field

- In tokamak configuration, there are extremums of magnetic field on a magnetic surface wehre $L_1 = \infty$.
- **Second-order non-uniformity**

$$B(z) = B_0 \left(1 + \frac{z}{L_1} + \frac{z^2}{L_2^2} \right)$$

- **Adiabatic motion**: $\epsilon = \dfrac{mv^2}{2}$, $\mu = \dfrac{mv_\perp^2}{2B}$ (Up to 4th order of $\tau = t - t'$)

$$z - z' = \sqrt{\frac{2}{m}} \sqrt{\epsilon - \mu B} \left[\tau - \frac{b}{3} \tau^3 \right] + (a + 2bz) \left[\frac{1}{2} \tau^2 - b \tau^4 \right]$$

$$a = \frac{\mu B_0}{m L_1}, \qquad b = \frac{\mu B_0}{m L_2^2}$$

- **Derivation of dielectric tensor is underway.**

Summary

- **Kinetic full wave analysis** using the **integral form of dielectric tensor** enables us to describe the **O-X-B mode conversion** of electron cyclotron waves.

- **Two-dimensional analysis on mid plane** in tokamak configuration (slab model) has shown the **spatial structure of O-mode, X-mode, and electron Bernstein wave**. It was confirmed that O-mode is efficiently converted to EBW near the **optimum injection angle**.

- **Two-dimensional analysis on poloidal cross section** requires the plasma dispersion kernel function including the **second-order non-uniformity** of the magnetic field strength along the field line. Formulation and implementation is under way.